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Magnetohydrodynamic Kinks in Astrophysics

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Three-dimensional magnetic kinks in nonresistive plasmas may be created and annihilated in pairs and conserve their homotopy properties during their lifetime. Such kinks could prove relevant to astrophysical, geophysical, or laboratory plasma problems. We describe magnetic kinks with one axis of rotational invariance analytically and graphically. As an example, we examine their relevance to the puzzle of the origin of galaxies.

1. WHAT IS A KINK?

Homotopy theory, introduced into modern physics in Wigner's theory of spin, has proved fruitful in general relativity (Finkelstein and Misner, 1959); in quantum field theory, where it gave rise to the concept of topological solitons; and in solid state physics, where it recently appeared helpful in classifying defects (Toulouse and Kleman, 1976) and analyzing phase transitions. Because of its purely topological nature, it allows the study of broad classes of field configurations at one time, including highly asymmetric ones, without the need of their specific analytic expression. It establishes new conservation laws not derivable from Noether's theorem.

Basic definitions and theorems of homotopy theory are recalled in Appendix A.

Consider a physical field configuration as a continuous map $f: X \to Y$ from a topological space X (the *domain* or base space) to another topological space Y (the *range* or image space). The set of all the field configurations f of given value on the boundary of the domain can be divided into homotopy

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classes. Together with a proper definition of addition, these classes constitute the (absolute) homotopy group of the field. Insofar as the passage of time is a homotopy, an initial field configuration whose boundary value is maintained fixed can only evolve within its homotopy class: *The homotopy invariants of the field are constants of motion.*

In virtue of boundary conditions, X is usually some m-dimensional sphere S^m.

This conservation law is useful only when the homotopy group of the field considered is nontrivial, that is, possesses more than one element. This nontriviality is a known property of the two topological spaces X, Y of the field in many cases. For instance, the homotopy group of an angle field ($Y = S^1$ = the circle) is nontrivial on a line ($X = S^1$) but is trivial on a plane ($X = S^2$). In the nontrivial cases, it follows that there exist certain field configurations that are not continuously transformable into the uniform field $f =$ const. They are called *kinked.*

Using the addition of the homotopy group we can make the most general class of field configurations out of a small number of special classes, the generators of the group. A field configuration belonging to one of these generating classes of the homotopy group is called a *kink* or *homotopon.* (Finkelstein, 1966).

Search for homotopons, when they are known to exist, has great heuristic value. It forces the study of new field configurations whose structures are richer and less symmetric than the standard ones and persist in time.

In this paper, we apply homotopy theory to the magnetic field of a nonresistive plasma. We establish the theoretical existence of kinks in such a field, give examples with one axis of rotational invariance, and consider their relevance to the problem of the formation of the galaxies.

2. EXISTENCE OF MAGNETOHYDRODYNAMIC KINKS

For a physical field to possess kinks, the appropriate homotopy group must be nontrivial, and some nontrivial configurations must be physical, that is, obey the specific constraints of the field, such as the existence of a corresponding energy-stress tensor in the case of a gravitational kink, or $\operatorname{div} B = 0$ for a magnetic kink.

2.1. The Homotopy Group of the Plasma Magnetic Field is Dynamically Nontrivial. In general, nontriviality may occur in two ways: as a kinematical property of the field, or as a dynamical one. The field is kinematically nontrivial if its domain S^m and its range Y define a nontrivial homotopy group: $\pi_m(Y) \neq 0$. This is the case for the gravitational tensor field, but *not* for the magnetic field, nor for any *kinematically linear* field, one whose range Y is a linear space homeomorphic to R^n. For them, every continuous mapping $f\colon X \to Y$ is homotopic to the zero field $f \equiv 0$ via the homotopy $Sf(0 \leqslant S \leqslant 1)$. The homotopy group of any kinematically linear field is therefore trivial.

Although Maxwell's equations are linear, the dynamical equations of the magnetic field in a plasma are not, and modify the topology of the range. In a nonresistive plasma [obeying the magnetohydrodynamic (MHD) equations] the flux of the magnetic field is carried with the plasma, trapped. As a consequence, if the magnetic field initially vanishes nowhere, it never vanishes anywhere. Thus the dynamics and certain initial conditions banish the point $B = 0$ from the image space, which becomes homeomorphic to $E^3 - 0$.

Does this modify the homotopy group? The domain being S^3, the relevant group is $\pi_3(E^3 - 0)$. This group is isomorphic to $\pi_3(S_2)$:

$$\pi_3(E^3 - 0) = \pi_3(I_1 \times S_2) = \pi_3(I_1) \times \pi_3(S_2) = \pi_3(S_2)$$

$\pi_3(S_2)$ is known to be the infinite cyclic group Z:

$$\pi_3(S_2) = Z$$

Thus, homotopy theory permits MHD kinks to exist, and predicts that they compose as integers add.

2.2. Physical MHD Kinks Exist. The only constraint imposed on a magnetic field by the MHD equations is $\operatorname{div} B = 0$. We will prove by construction the existence of kinks obeying this constraint.

In 1935 Hopf constructed a generator of $\pi_3(S^2)$ known as the Hopf map, which has historical significance: It established the difference between homology and homotopy. We need it in our construction.

A. The Hopf Map. In his construction, Hopf represents the domain S^3 by the set of unit spinors, pairs of complex numbers (z_1, z_2) such that $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$. The range S^2 is represented by the quotient space of this S^3 by the equivalence relation

$$(z_1, z_2) \sim (z_1', z_2') \Leftrightarrow (\exists \lambda \in C)\begin{cases} z_1 = \lambda z_1' \\ z_2 = \lambda z_2' \end{cases}$$

The equivalence class of (z_1, z_2) is designated by $(z_1 : z_2)$ and is called a ratio.

This quotient is in fact R^2 plus the point at infinity, $R^2 \cup \infty$, a homeomorph of S^2: Each equivalence class $(z_1 : z_2)$ with $z_2 \neq 0$ can be thought of as a point in R^2 [parametrized naturally by the representative $(z, 1)$; the point at infinity of R^2 is represented by the unique class $(1:0)$].

The Hopf map is defined as the natural map from the unit spinor space to the quotient space, namely, the map

$$H\colon (z_1, z_2) \to (z_1 : z_2)$$

$$\begin{tikzcd} S^3 \arrow[r, "H"] \arrow[dr, "P"] & R^2 \cup \infty \arrow[d, "\mathrm{St}_2"] \\ R^3 \cup \infty \arrow[u, "\mathrm{St}_3"] \arrow[r, "B"] & S^2 \end{tikzcd}$$

Fig. 1. Computation of the Hopf kink B.

B. Vector Kinks. As constructed by Hopf, H is a map from S^3 to $R^2 + \infty$. For a physical vector field, we need a map B between the respective homeomorphs $R^3 + \infty$ and S^2: If for convenience we designate by St_n the *stereographic projection* St_n: $R^n \to S^n$, we need to compute (Figure 1)

$$B = \mathrm{St}_2 \circ H \circ \mathrm{St}_3$$

(i) *The Spin Map:* $\mathrm{St}_2 \circ H$ is a map of spinors into unit 3-vectors. One such map arises naturally by associating to the state Ψ of a spin-$\frac{1}{2}$ object the expectation value of the spin vector in state Ψ:

$$P: \Psi \to \Psi^* \sigma \Psi$$

(σ are the three Pauli matrices). We call P the spin map. In Appendix B, we show that $\mathrm{St}_2 \circ H$ is precisely the spin map

$$\mathrm{St}_2 \circ H = P$$

(ii) *The Hopf Kink:* The composition $\mathrm{St}_3 \circ P$ associates with each space point $\mathbf{r} = r\mathbf{1}_r$ a vector obtained by applying a rotation $R(\mathbf{n}(\mathbf{r}), \omega(\mathbf{r}))$ with axis $\mathbf{n}(\mathbf{r})$ and angle $\omega(\mathbf{r})$ to a fixed vector, say $\mathbf{n}_z$.

We show that the axis of rotation is $\mathbf{n}_r$, the radial direction at $\mathbf{r}$, and the rotation angle $\omega(\mathbf{r})$ is a function of the radial distance r alone:

$$\omega(\mathbf{r}) = 4 \tan^{-1} r$$

Proof. (a) The stereographic projection St_3 can be conveniently considered as a map that associates with each space point $\mathbf{r}$ a unit spinor Ψ obtained by a rotation $U(\mathbf{r})$ of a fixed unit spinor Ψ_0:

$$\mathbf{r} \to \Psi = U(\mathbf{r})\Psi_0$$

with

$$U(\mathbf{r}) = \frac{i - \mathbf{r}\cdot\sigma}{i + \mathbf{r}\cdot\sigma}$$

$U(\mathbf{r})$ is unitary, connected to the identity, and maps the points at infinity into $-I$.

(b) It is well known that any element of $SU(2)$ can be written as $\exp(i\omega/2\mathbf{n}\cdot\boldsymbol{\sigma})$ with $\mathbf{n}$ a unit vector and ω a scalar. In Appendix C, we show that in our case

$$\mathbf{n} = \mathbf{1}_r, \qquad \omega = 4 \tan^{-1} r$$

(c) Furthermore, it is also well known that the above transformation of a spin-$\frac{1}{2}$ state induces a rotation with axis $\mathbf{n}$ and angle ω of the expectation value of the spin. B associates therefore with each space-point $\mathbf{r}$ the vector $B(\mathbf{r})$ obtained by applying $R(\mathbf{1}_r, \omega(\mathbf{r}))$ to a fixed vector, say $\mathbf{1}_z$. Since $\omega(\infty) = 2\pi$, it follows that $\mathbf{1}_z$ is the value of the field at infinity.

Let us write $n_\times$ for the linear operator on vectors defined by

$$(\mathbf{n}_\times)\mathbf{v} = \mathbf{n}_\times \mathbf{v}$$

Any rotation $R(\mathbf{n}, \omega)$ can be conveniently written as $e^{\omega\mathbf{n}\times}$. Indeed, $e^{\omega\mathbf{n}\times}$ is orthogonal, since $\mathbf{n}_\times$ is antisymmetric, and leaves $\mathbf{n}$ invariant. Thus the *Hopf kink*, the vector field corresponding to the Hopf map, is in this notation

$$B(\mathbf{r}) = e^{\omega\mathbf{n}\times}\mathbf{1}_z$$

(iii) *Other Vector Kinks:* Possessing one nontrivial field, we can now construct others by the following operations.

(1) Substitute for $\omega(r)$ any continuous scalar field that satisfies $\omega(0) = 0$ and $\omega(\infty) = 2\pi$.

(2) Multiply $B(\mathbf{r})$ obtained above by any continuous scalar field $A(\mathbf{r}) \neq 0$.

(3) Restrict the base space to a finite ball $r \leqslant r_0$ and require the same boundary condition for ω.

These do not change the homotopy class of the field.

We have constructed in this way a family of kinks

$$B(\mathbf{r}) = A(\mathbf{r})e^{\omega(r)\mathbf{1}_r\times}\mathbf{1}_z$$

with $\omega(0) = 0$ and $\omega(r_0) = 2\pi$. Do some of them obey div $\mathbf{B} = 0$?

C. MHD Kinks. In spherical coordinates, $B(\mathbf{r})$ is

$$B_r = A\cos\theta$$
$$B_\theta = -A\sin\theta\cos\omega$$
$$B_\varphi = -A\sin\theta\sin\omega$$

Let $A(r)$ be a function of r alone. Then the condition div $\mathbf{B} = 0$ is equivalent to

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2 A_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}A_\theta\sin\theta = \frac{\cos\theta}{r}\left[r\frac{dA}{dr} + 2A(1-\cos\omega)\right] = 0$$

The differential equation in $A(r)$

$$r\frac{dA}{dr} + 4\sin^2\frac{\omega}{2}A = 0$$

has a solution $A_\omega(r)$ provided $\omega(r)$ vanishes faster than $r^{1/2}$ in the vicinity of the origin:

$$A_\omega(r) = B_\infty \exp - 4 \int_\infty^r \frac{\sin^2 \frac{1}{2}\omega}{r}\, dr$$

B_∞ is the imposed boundary value.

Conclusion: Any field of the form

$$\begin{aligned} B(r) &= A_\omega(r) e^{1_r \omega(r) \times} \mathbf{1}_z \\ \omega(0) &= 0, \qquad \omega \gtrless r^{1/2} \qquad \text{for } r \to 0+ \\ \omega(r_0) &= 2n\pi \qquad \text{for } r \geqslant r_0 \\ n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

is a configuration of n kinks with no monopoles. The number n is the *homotopic charge* of the field.

3. INSTANTANEOUS KINEMATICS OF MHD HOMOTOPONS

3.1. Flux Lines of the Hopf Kink. The kinks of 2.2C all exhibit the same general pattern.

The equations of the lines of force are

$$\frac{dr}{dS} = \cos\theta$$

$$\frac{d\theta}{dS} = -\sin\theta \frac{\cos\omega}{r}$$

$$\frac{d\varphi}{dS} = \frac{-\sin\omega}{r}$$

A cross section of the flux surfaces by half a plane through the axis of symmetry is given in Figure 2; for computation the explicit expression of ω was taken to be $\omega(r) = 2\pi \sin(\pi r/2)$ $(0 \leqslant r \leqslant 1)$. As seen, the open lines of force fixed at the boundary lock in a toroid made of coaxial toroidal flux surfaces. The common equatorial axis is a circle (corresponding to $\omega = \pi/2$). Another equatorial circle rounds the external "edge" of the toroid ($\omega = 3\pi/2$).

Viewed from above the xy plane, the open lines (Figure 3) twist as they approach the origin.

Views from above of the toroid lines (Figures 4–6) reveal that they are poloidal–toroidal and that, in general, they ergodically fill the toroid surfaces.

Existence of Limit Cycle. The equatorial circular flux line at the outer edge of the toroid is a separatrix. Figure 7 shows the behavior of the flux lines in its vicinity as obtained by stability analysis. Together with the previous

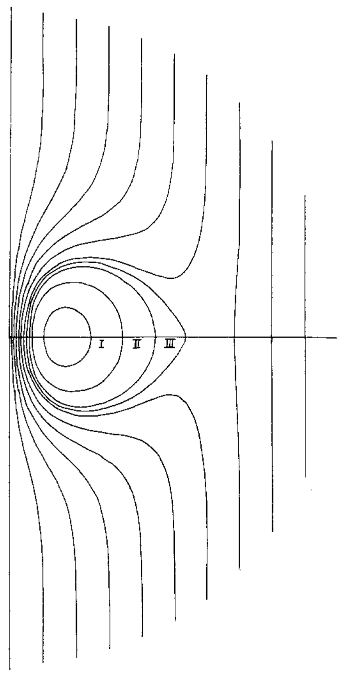

Fig. 2. Magnetohydrodynamic axisymmetric kink; cut view of flux surface.

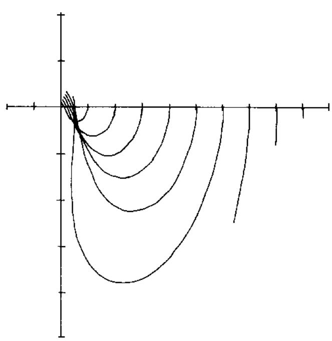

Fig. 3. View from above of open lines of force (above equator).

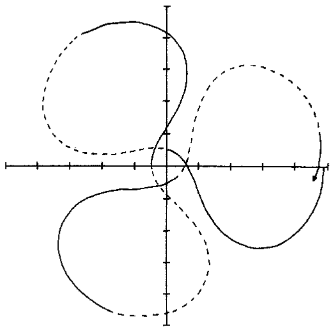

Fig. 4. View from above of line of force on flux surface III.

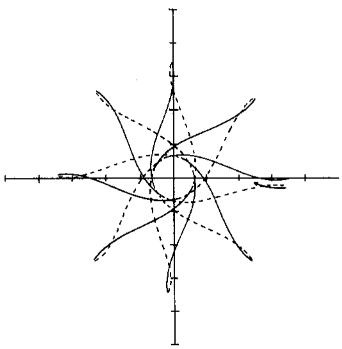

Fig. 5. View from above of line of force on flux surface II.

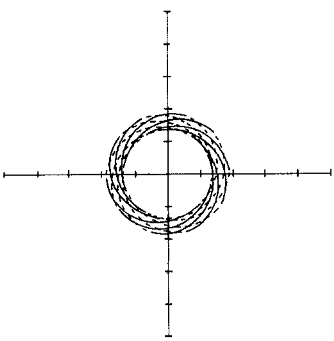

Fig. 6. View from above of line of force on flux surface I.

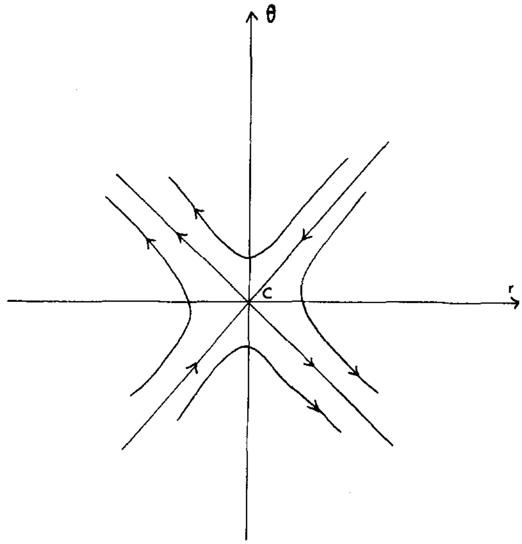

Fig. 7. Stability analysis around the "stationary point" $C(\theta = \frac{1}{2}\pi, r = \omega^{-1}(\frac{3}{2}\pi))$. The "time parameter" is the angle φ.

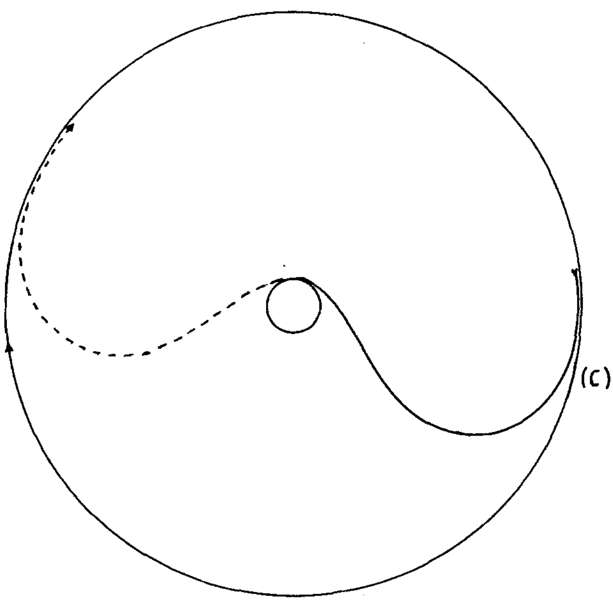

Fig. 8. View from above of a flux line on the toroidal surface with C as a limit cycle.

global views, it indicates that C is a *limit cycle* for two flux surfaces: the surface of the toroid (Figure 8), and one axisymmetric surface of open flux lines.

Toroidal magnetic configurations possessing limit cycles on their surfaces have been studied recently for their stability against global interchanges and other MHD instabilities (B. K. Harrison, et al., 1973).

3.2. Other Kinks. Homotopies of the Hopf kink give new kinks.

A. "Angular" Homotopies. Any function $\omega(r, \theta)$ that obeys:

$$\omega(r_0, \theta) = 2\pi$$
$$\omega(0) = 0$$

(where r_0 is the radius of the boundary) is homotopic to $\omega_c(r)$ (defined or satisfying $\omega_c(0) = 0$, $\omega_c(r_0) = 2\pi$) by the homotopy

$$\Omega(r, \theta, t) = (1 - t)\omega_c(r) + t\omega(r, \theta), \qquad 0 \leqslant t \leqslant 1$$

$e^{\mathbf{1}_r \Omega \times}\mathbf{1}_z$ is therefore a homotopy between $e^{\mathbf{1}_r \omega(r,\theta)\times}\mathbf{1}_z$ and $e^{\mathbf{1}_r \omega_c(r)\times}\mathbf{1}_z$. We call it "angular homotopy."

Conclusion: $A(r, \theta)e^{\mathbf{1}_r \omega(r,\theta)\times}\mathbf{1}_z$ are (new) magnetic kinks if and only if they are divergence free, i.e. if A satisfies

$$r \cos\theta \frac{\partial A}{\partial r} - \sin\theta \cos\omega \frac{\partial A}{\partial \theta} + A \sin\theta \sin\omega \frac{\partial \omega}{\partial \theta} + 2A\cos\theta(1 - \cos\omega) = 0$$

Angular homotopies may add pairs of antikinks to the initial kink.

B. "Spatial Homotopies." The Hopf kink associates with each space point an element of the rotation group $SO(3)$ in a one-to-one correspondence (the trivial rotations being mapped to the boundary). Thus, to any homeomorphism of the space leaving invariant the boundary corresponds a homeomorphism of the rotation group, which is a homotopy for the field. We call it "spatial homotopy."

A spatial homeomorphism as an invertible coordinate transformation, preserves flux lines and flux surfaces.

Conclusion: New kinks can be obtained by homeomorphisms of the flux surfaces of the Hopf kink.

4. DYNAMICS OF KINKS

As members of nontrivial homotopy classes, kinks have characteristic dynamical properties.

4.1. Homotopic Charge Conservation

Basic: Homotopic charge is a constant of motion. The configuration evolves within its homotopy class.

Inferences: This class, when nontrivial, is a limited region of functional phase space and may contain high-energy configurations which have *no mode of decay* to homotopically trivial lower-energy configurations. This leads one to seek solitonlike behavior, as well as other forms of nondissipative energy transport.

Whenever time evolution is a spatial homotopy, toroidal flux surfaces remain so and bound the plasma. The dimensional energy estimate $E = \Phi^2/R$ (where Φ is the flux through the torus, conserved; and R is the major radius) shows that R tends to grow.

Restriction: Homotopic charge may be created or destroyed when time evolution is not a homotopy, in particular when configurations possessing discontinuities occur, as in shocks. This important possibility can be checked by numerical computation.

4.2. Creation and Annihilation in Pairs

Basic: Magnetohydrodynamic homotopons combine like integers. In particular, they may be created in pairs ($1 - 1 = 0$): From a zero-homotopic-charge configuration may emerge two homotopons of opposite charge: a kink and an antikink.

Similarly, a kink–antikink pair may annihilate.

Inferences: A kink homotopic to the Hopf kink by an angular homotopy has the opposite $\omega(r, \theta)$ value of its antikink (equivalently, its opposite B_φ).

Pair creation seems the simplest mode of kink production. The most plausible environment for spontaneous pair creations is turbulence. The rate of production is expected to depend on the energy of the turbulence.

4.3. Topological Stability of Kinks. The kinks we constructed look roughly like flux rings in a vertical field. Is their topological stability more than a corollary to flux conservation? After all, *any* closed flux tube conserves its topology in a nonresistive medium. Since the lines of force are carried by the fluid, and the moving fluid particles remain in the neighborhood of each other, a closed line of force remains closed and a toroid evolves to a homeomorph.

However, the conservation law of kinks is stronger. Kinks are stable, toroids are not. For one topical example, in the presence of small but finite resistivity, lines of force are not restricted to follow exactly the fluid particles, the loops can open, and flux tubes can disappear immediately. For kinks, it is not flux conservation that is essential but *only that the field does not pass through the value zero*. For a plasma of finite resistivity σ and dimension L this assures a kink lifetime of about $4\pi\mu\sigma L^2/c^2$.

During that time, kinks can evolve in principle from a configuration including one toroid to a homotopic one with three toroids (see Section 4.2) since flux tubes are not preserved.

5. KINKS AND THE ORIGIN OF GALAXIES

The theory of the formation of the galaxies from cosmic hydrodynamic turbulence has been reintroduced by Ozernoi and Chernig (1968). In its new version it presents many attractive features and deals successfully with all the main problems but one: Does the cosmic turbulence decay catastrophically during the matter-dominated era before recombination? No satisfactory mechanism preventing the decay has yet been found.

We ask whether, in the process of magnetohydrodynamic turbulence, hydrodynamic turbulence decays but magnetic homotopons survive and shape the protogalaxies. We ask indulgence for a looseness of discussion inevitable at the present stage of knowledge.

5.1. Present State of Cosmic Turbulence Theory. Ozernoi and Chernin (1968) exploited von Weizsäcker (1948) and Gamow's (1952) cosmic turbulence theory within the framework of the "hot big bang" cosmology (see recent review by Jones, 1976). They suppose the existence of a primeval strong hydrodynamic turbulence and postulate a *turbulence scale*, that is, a largest scale whose dynamical time scale is equal to the cosmic expansion time scale. They distinguish between three successive regimes:

A. During the radiation-dominated era ($t < t_{eq}$) the turbulence scale increases with time: The larger scales successively encompassed in the turbulence provide a source of energy against decay.

B. During the matter-dominated era, before recombination ($t_{eq} < t < t_{rec}$) the scale of the turbulence decreases with time. Estimates of the mass associated with the largest turbulent eddy M are of galactic size.

C. After recombination ($t > t_{rec}$) the sound speed becomes much lower than the characteristic velocities. Supersonic motions generate large density fluctuations of scale M. Because of the cosmic expansion, the potential energy finally exceeds the kinetic energy and M condenses to form a galaxy.

5.2. Unsolved Issues of the Cosmic Turbulence Theory. During the second phase ($t_{eq} < t < t_{rec}$), as the scale of the turbulence decreases, support from larger scales against viscous decay is considerably reduced. It is an unproven assumption of the theory that the mean straining rate due to the frozen eddies is sufficient to maintain the turbulence. Jones (1973) showed that if free decay is allowed, all the turbulence would decay prior to recombination unless $t_{eq} \sim t_{rec}$.

Peebles (1971) pointed out that during the third phase ($t > t_{rec}$), because of the existence of matter currents whose hydrodynamic time scale are much shorter than the cosmic expansion time scale, shock waves would compress a large fraction of the matter in the universe into small, dense, bound lumps, which can hardly be identified with present galaxies.

5.3. Can Hydromagnetic Kinks Be Seeds of Present Galaxies? The existence of galactic (if not intergalactic) magnetic fields is now firmly established. Their primeval nature is probable, their origin still mysterious. They have perhaps been given less attention than they deserve, the general trend being to consider their role in cosmic dynamics as secondary. Zel'dovich (1969), however, studied the possibility that inhomogeneous magnetic fields produce an inhomogeneous mass density of galactic size, and showed that this was compatible with present magnetic field values.

Here we consider the role of the cosmic magnetic field in cosmic MHD turbulence. Suppose magnetic kinks were present during the expansion. Can we use them in a plausible scenario?

Galaxies as Remnants of MHD Kinks: Consider the following speculation: After the first second, the universe is an expanding hydrogen helium plasma. Its resistivity is negligible. From a primeval global magnetic seed, turbulence generates local magnetic fields of high intensity, with correlation lengths of the order of the velocity correlation length, following a mechanism described by Batchelor (1950) for stationary MHD turbulence. In the background of the uniform field, homotopons spontaneously appear in pairs. Kinks and antikinks are produced and annihilated strongly in pairs, and weakly one by one, due to approximate conservation of homotopic charge. During the entire radiation-dominated era, turbulence is sustained by an increasing turbulent scale (regime A of Section 5.1).

When matter density exceeds radiation density, the regime changes, the turbulence scale decreases, and kink production slows. Except for large-scale "frozen out" eddies, the turbulence soon completely decays. Since isolated homotopons decay only weakly, they remain in the expanding universe. They confine matter in their toroidal flux surfaces and impart to the conducting matter an angular momentum, the germ of galactic spin.

The confined mass is typically the mass of the largest turbulent scale, M, of galactic size.

As the universe expands, gravity replaces magnetism in shaping the galaxy, for several reasons:

(1) During the expansion, the magnetic pressure decreases as $1/r^4$ (the flux being conserved), whereas the gravity pressure on the surface of confinement decreases only as $1/r^3$. Galaxies start to collapse.

(2) At $z = 1000$ the cosmic plasma begins to recombine. It takes 20% of the cosmic expansion time to reach near total recombination.

(3) The ordered structure of the homotopons may disintegrate in shock waves. Their present survival is not excluded, however. The interstellar magnetic arches observed in the neighboring part of our galaxy (Mathewson, 1968) are suggestive and require further study.

Conclusion: This speculation shows the new directions of research that this topological approach points out in continuum mechanics. Wherever the conditions of validity of the new conservation law hold (as in the sun, the earth core, or hot laboratory plasmas) homotopons might arise as structures of anomalous behavior and lifetime. One can pursue a similar analysis in hydrodynamics and show the existence of physical homotopons in the vorticity field of an invicite fluid.

Questions: Do stationary kinks exist? (Not in the class of configurations of Section 2.2C.) What is the rate of kink and kink-pair production? Can they be seen experimentally?

APPENDIX A: BASICS OF HOMOTOPY THEORY[2]

Homotopy classifies maps between two given topological spaces X and Φ.

A.1. Homotopy Relation. Consider all the continuous maps $\{g_i\}$ from X to Φ subjected to the condition at the boundary ∂X:

$$g(\partial X) = y_0$$

(y_0 is a fixed point in Φ).

Define the relation "homotopic to" ($\sim$):

$$g_0 \sim g_1 \Leftrightarrow \exists G(x,t) \begin{cases} G\colon X \times I \to \Phi & (I \text{ is the unit interval}) \\ G(x,0) = g_0 & \\ G(x,1) = g_1 & \\ G(\partial X, t) = y_0 & 0 < t < 1 \end{cases}$$

Loosely, g_0 is homotopic to g_1 if there exists a continuous transformation of g_0 into g_1 which respects the boundary condition.

A.2. Homotopy Class. Theorem: $\sim$ is an equivalence relation.

The equivalence classes defined by the homotopy relation are called *homotopy classes.*

[2] See for instance Hilton 1944.

A.3. Homotopy Group. Homotopy classes 1 and 2 can be added if an addition of two representatives is defined consistently:

$$g_1 + g_2 := \begin{cases} g_1(2x), & 0 \leqslant x \leqslant \frac{1}{2} \\ g_2(2x-1), & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

$g_3 := g_1 + g_2$ belongs to class 3 (written $[g_3]$).

Theorem. Class 3 is independent of the choice of the representatives g_1 and g_2. This enables the definition

$$[g_1] + [g_2] := [g_1 + g_2]$$

Loosely, the representative of the sum of two classes is obtained by adding two representatives whose nontrivial structures do not overlap.

Theorem. Taken with this addition, the classes form a group, the *homotopy group.*

If X is the Euclidean n-cube this group is denoted $\pi_n(\Phi)$.
Notation:

$$S^n = n\text{-sphere} = \left\{x_1, \ldots, x_{n+1} \,\middle|\, \sum_{i=1}^{n+1} x_1{}^2 = 1\right\}$$

$$E^n = n\text{-ball} = \left\{x_1, \ldots, x_n \,\middle|\, \sum_{i=1}^{n} x_i{}^2 \leqslant 1\right\}$$

Relevant homotopy groups:

$$\pi_1(S^2) = 1$$

$$\pi_2(S^2) = Z$$

$$\pi_3(S^2) = Z$$

1 means the group with one element, and Z means the group of the integers.

APPENDIX B: $\mathrm{H} \circ \mathrm{St}_2 = P$

The geometrical construction of the stereographic projection St_2 of $R^2 + \infty$ onto S^2 is shown in Figure 9.

If we parametrize $R^2 = \{(x, y)\}$ by $\{z = x + iy\}$, the image $\vec{\mathrm{B}}(z)$ by St_2 is

$$\vec{\mathrm{B}}(z) = \lambda(\rho 1_\rho - 1_z) + (1 - \lambda)1_z$$

(see Figure 9). ρ is the absolute value of z, 1_ρ its radial direction in the plane.

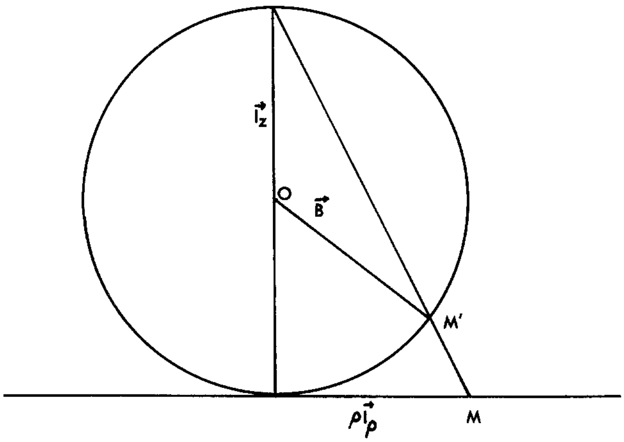

Fig. 9. Geometric construction of the stereographic projection of R^2.

λ is determined by the condition $B^2 = 1$, which yields

$$\lambda = \frac{4}{4 + \rho^2}$$

The Hopf map associates to the unit spinors $\Psi = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ the ratios z_1/z_2 which parametrize $R^2 + \infty$.

We take this parametrization to be

$$z = 2\frac{z_1}{z_2}$$

Then

$$\lambda = \frac{1}{1 + z_1\bar{z}_1/z_2\bar{z}_2} = z_2\bar{z}_2$$

and

$$\begin{aligned}\vec{B}(\Psi) &= (z_2\bar{z}_2 - z_1\bar{z}_1)1_z + 2|\bar{z}_1 z_2|1_\rho \\ &= \bar{\Psi}\sigma_z\Psi 1_z + \bar{\Psi}(|\sigma_x + i\sigma_y|)\Psi 1_\rho \\ &= \bar{\Psi}\boldsymbol{\sigma}\Psi\end{aligned}$$

APPENDIX C

The computation of ω and $\mathbf{n}$ is simpler in quaternions: If the vector $\mathbf{r}$ is considered as an imaginary quaternion and the spinor Ψ as a unit quaternion, St_3 can be written

$$\mathbf{r} \to \Psi = \frac{1 + \mathbf{r}}{1 - \mathbf{r}}\Psi_0$$

It follows (if we take Ψ_0 to be 1)

$$\mathbf{r} = \frac{\Psi - 1}{\Psi + 1}$$

$$= \frac{e^{\ln \Psi} - 1}{e^{\ln \Psi} + 1} + \frac{e^{\ln \Psi/2} - e^{-\ln \Psi/2}}{e^{\ln \Psi/2} + e^{-\ln \Psi/2}}$$

$$= \tanh \tfrac{1}{2} \ln \Psi$$

Therefore

$$\Psi = e^{2 \tanh^{-1} \mathbf{r}}$$

If we write r as $|r|\mathbf{i}_r$ (with $\mathbf{i}_r^2 = -1$), then

$$\Psi = e^{2\mathbf{i}_r \tan^{-1} \mathbf{r}}$$

In spinor and vectorial language, this becomes

$$\Psi = e^{2\mathbf{1}_r \cdot \boldsymbol{\sigma} \tan^{-1} \mathbf{r}}$$

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